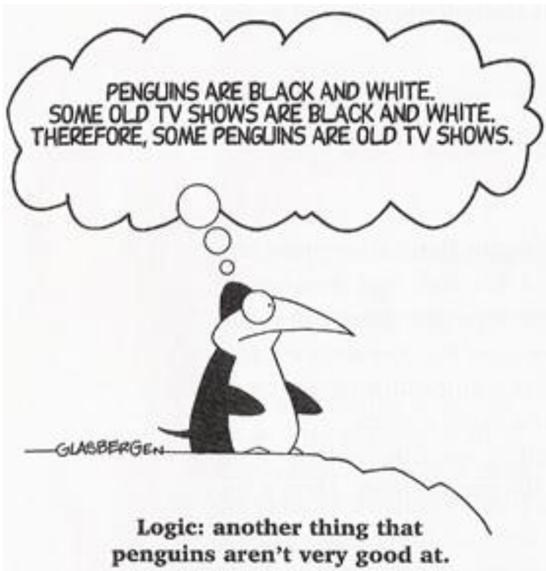
1. TRUTH IN MATHEMATICS

§1.1. What Is Truth?

"What I like about mathematics," said a student of mine once, "is that, when something is true, you can know for sure that it's true". He was comparing this subject with history where 'facts' are constantly challenged and interpretations are numerous.

It is true that there is remarkable agreement among mathematicians around the world about the truth of mathematical statements. Once a theorem is proved it's universally accepted as fact. Occasionally a proof has been challenged and subsequently proved to have a flaw, though in most cases the result ended up being true but just needed a sounder proof.



On rare occasions a published mathematical result has actually proven to be false. But once that has been pointed out the mathematical community has agreed that it's false and there the matter has rested.

§1.2. Mathematical Truth

Truth in a court of law refers to something that actually happened. But truth in mathematics refers to something more universal. A true statement about triangles not only refers to all the triangles you've ever seen, but also to all the triangles that can ever be drawn. This type of truth is predictive.

Consider the following areas of intellectual endeavour: mathematics, physics, psychology, history and philosophy. Each claims to make true statements, yet the methods for establishing these truths are quite different. And, before I elaborate, I feel the need to point out that I'm not making any inferences about the reasoning abilities of the mathematicians, physicists, psychologists, historians and philosophers. If I say that historians are less rigorous than mathematicians I'm referring to the nature of their subjects. It's impossible for a psychologist to be as rigorous as a mathematician simply because of the nature of psychology.

Let me begin with physics. Physics primarily employs the scientific method to establish its truths. A hypothesis is made, experiments are carried out, and the results either support the hypothesis or contradict it.

At least that's the case with experimental physics. Theoretical physics uses a lot of mathematics in addition to the facts that have arisen by experimentation. Frequently, in theoretical physics, predictions have been made by this mathematical activity which later, sometimes a lot later, have been verified experimentally. Perhaps I should make a territorial claim and insist that theoretical physics should be considered to be a branch of mathematics. But fundamentally physics is about the real world and all truth in physics rests on the foundation of experiment.

Physicists tend to agree on most things. Of course there are fundamental issues such as, "is light a series of particles, or a wave?" Here most physicists have given up on deciding and have accepted the answer "both" as paradoxical as it might seem. Where a few differences exist is in the realm of theoretical physics. If a theory has been developed using mathematics and has yet to be verified experimentally, there's room for disagreement. Of course you can't blame the mathematics! What will be argued over is the way the mathematics has been employed.

Psychology also employs the scientific method, but relies heavily on statistics. Incidentally I consider that the part of that subject that comes under the heading of Mathematical Statistics is also a branch of mathematics.

But I'm fine with it being studied in a different university department.

Experimental psychology involves experiments, mostly with human individuals. Compared to physics the outcomes are not so clear cut. If 99 out of 100 cases in a physics experiment support the hypothesis the conclusion would be that something went wrong with that one case and the conclusion would be that the hypothesis has been verified. But if only 95 out of the 100 cases supported the hypothesis the conclusion would be that there must be some extra factor that hadn't been taken into account. Perhaps those five cases were carried out on a particularly cold day and temperature might have to be included in the hypothesis.

By contrast a psychologist would be delighted by an experiment where 95 out of 100 cases supported the hypothesis. But what if only 60% were consistent with the hypothesis? Out comes statistics. The psychologist employs an appropriate statistical technique. This would calculate the probability that an outcome of 60%, or greater, could have occurred randomly where you would expect, on average, a 50% result. This would have to take into account sample size. If the sample size was a million, common sense tells us that a 60% outcome couldn't have arisen purely by chance. But sample sizes are usually much less than this.

The rule of thumb in most psychology experiments, and indeed in most areas that employ statistical methods, is that if the probability of such an outcome by chance is

5%, or less, then the hypothesis is considered to be verified.

There are parts of psychological research that don’t employ experimentation and these can be considered along the same lines as the philosophers.

Historians don’t have the luxury of being able to carry out experiments. “Let’s create 100 Henry VIII’s and measure what percentage of wives they execute!” Historians are concerned with specific events. They can only rely on written documents. But they don’t simply believe what they read. Often inferences have to be made from what has been recorded. Sometimes there are conflicts between documents. Historians don’t simply say “three out of five records say that so and so was murdered, so that must have been the case”. They examine the conflicting documents carefully and decide how reliable they are, given the testimony of other documents.

Biblical scholars use similar techniques. We have none of the original manuscripts for any part of the Bible, but we have numerous copies of copies that date three or four centuries later. These copies often differ – not a lot but often. Mistakes, deliberate or otherwise, occur when you have copies of copies and copies of copies of copies. Biblical scholars use non-mathematical techniques for reconstructing the best approximation to the original manuscript. They also employ scientific technology in dating the existing manuscripts. But they can’t be said to employ the scientific method.

Philosophers, by and large, can't carry out experiments and nor do they make use of physical evidence. A philosopher can carry out research in an arm-chair. He or she analyses common human experiences and uses reasoning to draw conclusions. Not surprisingly there are lots of schools of thought in philosophy. Historians don't disagree as widely from one another but there are still disagreements on many points.

But when you come to mathematics there's essentially no disagreement at all. Occasionally a false mathematical statement has been published, along with, of course, an invalid proof. But these are quickly challenged and shown to be faulty. A little more often a true statement is published with an invalid proof. Usually someone comes along and patches up the proof.

So do all mathematicians agree on all mathematical truth? Almost. The basic process of developing a mathematical theory in modern times is to present a sequence of definitions, theorems and proofs. A **definition** is where one mathematical concept is defined in more primitive terms.

Proofs are sequences of mathematical statements set in a logical framework and are a bit like a computer program. But what's important to realise is that you can't prove something from nothing. You have to begin with certain assumptions that are not proved. These are called **axioms**. Sometimes they are regarded as "intuitively obvious" but intuition can be unreliable.

It's interesting to contemplate that this situation is no different to theology. Religious belief is often derided because the followers accept statements, such as the existence of God, without any proof. One can assert that one intuitively knows that God exists. One can point to the remarkable things that have been done by people with a religious belief. But none of this evidence is considered convincing. Fundamentally the basic truths of a religious belief have to be accepted as axioms.

It's no different with mathematics. There is such a thing as a mathematical creed! The remarkable thing is, and this is where mathematics is very different to theology, most mathematicians accept these creeds, or ones that are logically equivalent to them. There are very few exceptions.

But there is a small number of axioms where there is not universal agreement. The most important of these is the Axiom of Choice. It's accepted by some mathematicians and rejected by others. But don't think that it's called the Axiom of Choice because one has the choice of accepting or denying it. It's because it makes a certain claim about the possibility of making infinitely many choices.

Actually, mathematicians tend not to divide into two opposing sects, as might be the case if mathematics was a religion. They're more pragmatic than that. Generally they try to prove theorems without using the Axiom of Choice but, if that's not possible, they're happy to use it and they simply note the fact that the proof relies on that axiom.

So the paradigm for mathematical truth is to start with a set of axioms and for the theory to consist of proofs that use only these axioms, definitions and, of course, previously proved theorems.

This sounds a wonderful way of doing things but it must be pointed out that it's been shown that this axiomatic approach has in-built limitations. Gödel has shown that any axiomatic system of any complexity – certainly any that includes basic arithmetic – has true statements that can't be proved from the axioms! When you consider that there are such things as undecidable statements in set theory (statements where it can be proved that they can never be proved true from the axioms but also that they can never be proved false) you begin to see that mathematics is not as logically pure as my student seemed to think it is!

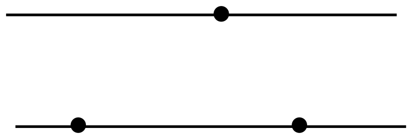
In these notes I'll be sketching how mathematics can be built up from a single set of axioms, and many definitions.

§1.3. The Disembodied Angel

Years ago one of my colleagues at Macquarie University, Alan Macintosh, invented a pedagogical tool called *The Disembodied Angel*. He died a long time ago and is probably now a disembodied angel himself.

The disembodied angel is an imaginary creature who is highly intelligent but who has no spatial sense. It lives in a spiritual realm and has no concept of geometrical entities.

Alan had a pair of walkie talkies (these days we'd use mobile phones). One student went into another room with one of the walkie talkies and played the part of the disembodied angel. He had to pretend that he had no geometrical intuition. Another student went to the board, in the lecture room, and tried to describe the following geometric construction to the angel in the other room.



Student: Well you've got two points which lie on a line.

Angel: I understand everything, except 'point', 'line' and 'lies on'.

Student: Well a point is like a dot.

Angel: I've never encountered a dot. Is it like a cherubim?

Student: No, it's something that has no length or breadth.

Angel: I don't know what 'length' and 'breadth' mean. What about a line? What exactly is it?

Student: It's something that's infinitely long but has zero breadth.

Angel: Oh, I understand 'infinite'. God is infinite. And I'm good with numbers, three for the Trinity, you know. Zero? Yes I remember a newcomer to Heaven once asking the Archangel Gabriel how many sins I had committed and Gabriel said, "zero". I think 'zero' is another way of saying "none".

Student: Yes that's right. We're getting somewhere at last.

Angel: But I still don't know about 'length' and 'breadth'.

Student: Well never mind. Just accept points and lines as undefined entities.

Angel: Fine, but what about 'lie on'. It sounds like some sort of relation.

Student: Yes, there's an undefined relation of a point lying on a line.

Angel: I'm fine with that too. Can a point lie on more than one line?

Student: Oh yes, all the time. Now you take a third point that doesn't lie on this line.

Angel: Got it.

Student: Well Euclid says there's exactly one line that passes through that third point and is parallel to the first line.

Angel: I presume Euclid is your friend. And I'm OK with 'exactly one'. I know that there is exactly one God. But I'm puzzled by 'passing through'.

Student: Oh that's easy. To say that a line passes through a point is just another way of saying that the point lies on the line.

Angel: Great. So all I need now is to understand 'parallel'. Is it another undefined relation, this time a relation between two lines?

Student: No. Two lines are parallel if they remain a constant distance from one another.

Angel: Distance?

Student: Well, the angle between two parallel lines is zero.

Angel: Do you mean the *angel* between the two lines?

Student (*starting to become frustrated*): Let me put it another way. The two lines don't meet.

Angel: Meet? I've heard of that in the mass. "It is right and meet so to do."

Student: It's nothing like that. How can I put it? There is no point that lies on both lines.

Angel: Oh now I get it. And that statement you said at the beginning. Is it true?

Student: Euclid says so. It's one of his postulates.

The point of this story is to explain that the modern abstract approach to mathematics is to set up an axiomatic creed – undefined entities, definitions, and axioms. The subject development follows by using logic to prove theorems. It makes no use of intuition and should be intelligible to a disembodied angel.

That's not to say that intuition plays no role in mathematics, far from it. A good mathematical exposition should make use of the reader's intuition to help him or her understand the formal proof. But, at least with advanced students, the formal proof should be capable of standing alone.

Mathematical intuition is in fact what drives mathematical research. No mathematician ever discovers his or her theorems by playing with axioms. He or she finds them by use of a highly developed intuition. But as valuable as that intuition is, a mathematician doesn't stop until they can translate their proof into a formal series of deductions that can stand alone without their intuition.

There was once an Indian called Ramanujan who amazed some Oxford professors with his intuition in the area of infinite series. You may have seen his story in the movie *The Man Who Knew Infinity*.

He knew intuitively a huge number of new mathematical results but he had a poorly developed concept of proof. He claimed it was an Indian goddess who revealed these mathematical truths to him. But whenever he announced some new result, the professors went away and could always come up with a proof.

The tool for creating proofs is logic, and logic itself has many levels of complexity. Here we'll take a naïve view of logic, which is what the vast majority of mathematicians do, and leave the logical complexities to logicians. But even naïve logic isn't as straightforward as one might think. There are logical pitfalls to be avoided and it doesn't seem possible to give clear-cut rules that guarantee that we will always avoid such pitfalls!

Let me warn you that the next few chapters will be hard going. As we set up the complex number system you'll probably get frustrated because you already know all that stuff. By all means skip the rigorous development and rely on the intuition that seemed to satisfy you at school. But if you do skip these proofs you must *never* say that one should only believe what one can prove!

Perhaps you might say, “I don’t have to check the proofs myself. Dr Cooper has provided the proofs and he’s a reliable source.” That would be even worse. It would be like believing in God just because that’s what it says in the Bible!